Appendix F
The Effect of Cycle Slips on Recovered Refractivity

Equation (A-28) gives the error in the recovered refractivity expressed in terms of the error profile in the bending angle. We use that expression to estimate the error in recovered refractivity that results from a "flywheeling" receiver failing to account for a transient in excess Doppler and, hence, bending angle. To simplify the analysis, we consider the case of a ray encountering a hard discontinuity ΔN at a radius r_o; this is the Case A scenario discussed in the main text. Also, Fig. C-2 shows examples of the transient Doppler profile for this case. To further simplify the analysis, we set the reference refractivity profile to zero. Therefore, we need only the Snell term $\Delta\alpha_S$ for the transient incurred when the main ray hits the boundary bearing the discontinuity. From Eq. (A-48), $\Delta\alpha_S$ may be written in the form

$$\Delta\alpha_S \doteq 2\sqrt{2}\left[\sqrt{\nu} - \sqrt{\nu + \Delta N}\right], \quad \nu = \frac{r_o - r}{r_o} \tag{F-1}$$

Let us consider the case where $\Delta N = N^+ - N^- < 0$, which leads to a shadow zone where receiver operations might be difficult. In this case, we assume that the receiver fails to "track" the excess Doppler associated with this discontinuity; thus, the phase error made by the receiver results in an excess Doppler profile that is equivalent to a bending angle $-\Delta\alpha_S(a)$. If we assume that the receiver fails to track any portion of the transient from its onset at the boundary down to an impact parameter value a, then it follows from Eq. (A-28) that

$$\delta\hat{N}(a) = -\frac{1}{\pi}\int_{a}^{\tilde{a}} \frac{\Delta\alpha_S(\xi)}{\sqrt{\xi^2 - a^2}}\, d\xi \tag{F-2}$$

where $\tilde{a}$ is the impact parameter at the critical refraction altitude $\tilde{a} = r_o(1+\Delta N)$. If we insert $\Delta\alpha_S$ from Eq. (F-1) into Eq. (F-2) and make some simple but accurate approximations that exploit the large magnitude of a, it can be shown that

$$\delta\hat{N}(\nu) \doteq \Delta N - \frac{2}{\pi}\left(\nu \sin^{-1}\left[\sqrt{\frac{-\Delta N}{\nu}}\right] - \sqrt{-\Delta N(\nu + \Delta N)}\right) \tag{F-3}$$

If we let $\nu \to \infty$, we obtain

$$\delta\hat{N} \to \Delta N \tag{F-4}$$

The recovered refractivity asymptotically assumes a bias equal to the discontinuity, a not too surprising result, and a negative bias in this case.

The reader may object on the grounds that the receiver almost surely would be able to track part of the transient, particularly after the signal returns upon exiting the shadow zone. Suppose that the receiver resumes normal operations at the flaring point associated with the first contact with the caustic (the significant interference at this point notwithstanding; see, for example, Fig. 2-11) and continues to "track" error free at all altitudes below this point. At the contact point, we have from the thin-screen model and Eq. (2.5-1) that $dh_{\mathrm{LG}} / dh = 0$. From Eq. (F-1), it follows that the value of ν at this point is given by the condition

$$\frac{2D}{r_o}\left(\nu^{-1/2} - (\nu + \Delta N)^{-1/2}\right) = 1 \tag{F-5a}$$

where D is the limb distance. When $\nu >> |\Delta N|$, it follows upon expanding Eq. (F-5a) in powers of ν that at the contact point

$$\nu^{\dagger} \doteq \frac{1}{2}\left(-2\Delta N \frac{D}{r_o}\right)^{2/3} \tag{F-5b}$$

which is indeed very much larger than $|\Delta N|$ for small $|\Delta N|$. Hence, the asymptotic form for $\delta\hat{N}$ in Eq. (F-4) still applies even though the receiver is able to partially recover.

In summary, most of the error from cycle slips is incurred in the shadow zone for this example, and the recovered refractivity for subsequent altitudes below this point suffers a negative bias of ΔN. One might expect to find cycle slips of opposite polarity associated with scenarios where $\Delta N > 0$. Although a ΔN of positive polarity can produce significant interference, it does not cause shadow zones. Also, we note from Figs. 2-11, 3-23, and 3-24 that the "tracking-loop stress" at and just below the boundary (i.e., phase acceleration, which would result from a mis-modeled phase profile in the receiver software, or equivalently, the difference between the extrapolated Fresnel phase from the main ray and the actual Fresnel phase in the shadow zone) differs sharply, depending on the polarity and magnitude of ΔN, with the negative polarity showing a larger phase windup because of the critical refraction or super-refracting situation that can occur for this polarity. In a true shadow zone, there are no rays arriving at the low Earth orbiting (LEO) satellite and, therefore, no stationary phase path to be tracked, which provides a limiting scenario that will be approached for large gradients in refractivity near the boundary. There will be a hiatus in tracking for such zones. On the other hand, when $\Delta N > 0$, there is no hiatus; there is at all points at least one stationary phase path to be "tracked" well before the tangency point of the main ray reaches the boundary. The challenge for the analyst in this case is to assign the correct phase and amplitude to each tone in this triplet zone (see Fig. 2-2a) when there is abundant signal power but significant interference.

We can obtain a rough estimate of the potential cycle slipping in a shadow zone by evaluating the difference between the stationary value of the Fresnel phase at a given epoch and the extrapolated phase that would be observed at the same epoch from the main ray. The latter would be the observed phase produced by the reference phase profile in the thin screen. Let us choose as the epoch the first contact with the caustic where the signal power will be near its maximum value. The extrapolated phase from the main ray would be zero in this example because we have set the thin-screen reference phase profile to zero to simplify the analysis. From Eqs.(2.5-1), (2.8-3), (F-1), and (F-5b) it follows that the stationary value of the Fresnel phase at first contact with the caustic is given by

$$\Phi\left(h^{\dagger}, h_{\mathrm{LG}}^{\dagger}\right) = \frac{2^{5/2}}{3} k r_o (-\Delta N)^{4/3} \left(3 \left(\frac{D}{16 r_o} \right)^{1/3} - (-\Delta N)^{1/6} \right) \tag{F-6}$$

The point of actual maximum in signal-to-noise ratio (SNR) is slightly offset from $h_{\mathrm{LG}}^{\dagger}$, which is located at $\hat{h}_{\mathrm{LG}}$. The correction to the geometric optics prediction for the point of maximum flaring is given in Appendix D. From Eq. (D-7), it follows that

$$\hat{h}_{\mathrm{LG}} - h_{\mathrm{LG}}^{\dagger} = 1.47 \left[\frac{\lambda^2}{4\pi^2} \left(\frac{D^4}{(-\Delta N)^2 r_o} \right)^{1/3} \right]^{1/3} \tag{F-7}$$

and one could as easily evaluate the Fresnel phase at $\hat{h}_{\mathrm{LG}}$ instead of at $h_{\mathrm{LG}}^{\dagger}$. At any rate, Eq. (F-6) gives for the Fresnel phase a value of about 1/2 cycle for $\Delta N = -1 \times 10^{-6}$, and 10 cycles for $\Delta N = -1 \times 10^{-5}$, which are the same levels of cycle slipping seen in Figs. 2-11, 3-23, and 3-24.

A "smart" receiver could be characterized by its ability to account for adverse signal conditions, i.e., multiple tones, interference, and low SNR, and by its capability for near-real-time backward signal recovery and phase connection, etc. Whether these capabilities are found in near-real-time data products from an advanced receiver design or obtained in non-real time from advanced (e.g., Fourier) data analysis techniques, is immaterial. In either case, over a number of trials, one would expect even "smart" designs to have more difficulty with shadow zones than with strong interference zones. This would result statistically in a tendency to incur a net loss of cycles over a series of layers presenting adverse signal conditions involving both polarities for ΔN. If this surmise is accurate, there would be a tendency for recovered refractivity profiles to be negatively biased in the lower troposphere for setting occultations when one cannot reliably deal with adverse signal conditions either by smart receiver design or by advanced techniques for signal recovery.